Two-Dimensional Classification of 3x3 Symmetric Games Under Replicator Dynamics

All information from: Immanuel M. Bomze's "Lotka-Volterra equation and replicator dynamics: A two-dimensional classification" (1983) and "Lotka-Volterra equation and replicator dynamics: New issues in classification" (1995)

Replicator dynamics (RD) were introduced by Taylor and Jonker (1978) to model evolution of behaviour in intraspecific conflicts under random pairwise interaction in a large, ideally infinite population. It is used for modeling many biological processes including the evolution of animal behaviour, selection in population genetics, and prebiotic evolution.

RD formalizes the idea that the growth rates $\dot{x}_i/x_i$ of relative frequency x_i of the i^{th} behaviour pattern ($i=1,\ldots,n$) is equal to the (dis)advantage

$$e_i \bullet Ax - x \bullet Ax = \sum_j a_{ij}x_j - \sum_{j,k} x_k a_j x_j$$

measured by incremental fitness relative to the average performance within the population in state $x = [x_1,\ldots,x_n]$. Here a_{ij} denotes incremental individual fitness attributed to an i-individual when encountering a j-individual, and $A = [a_{ij}]$ is the resulting fitness matrix. A dot $\dot{}$ denotes derivative w.r.t. time t.

This infographic presents the case of $n = 3$ behaviour patterns yielding the system of cubic differential equations

$$\dot{x}_i = x_i \sum_j \left[a_{ij} - \sum_k x_k a_{kj}\right] x_j, \quad i=1,2,3 \qquad \text{(RD)}$$

operating on the state space $S=S^3$, where for general n

$$S^n = \{x = [x_1,\ldots,x_n] : x_i \geq 0, \text{ for all } i,\ \sum_i x_i = 1\}$$

denotes the standard simplex in n-dimensional Euclidean space.

The complete list of possible phase portraits under the replicator dynamics is presented bellow, containing 49 qualitatively different cases up to flow reversal: 19 robust ones and 30 non-robust. [proof in Bomze (1983, 1985)].

Here, for a phase portrait to be *robust* means that it is invariant under sufficiently small perturbations of fitness parameters.

Given our classification, we can see how the game theoretic solution concept of evolutionarily stable sets (ES sets) [introduced by Thomas (1985)] fairs in capturing the asymptotic behaviour under RD of this class of games.

Formally, a set of states P is said to be an ES set *iff* for every $p \in P$ we have

$$x \bullet Ap \leq p \bullet Ap \qquad \text{for all } x \in S^n, \text{ and}$$
$$x \bullet Ax < p \bullet Ax, \qquad \text{if } x \notin P \text{ with } x \bullet Ap = p \bullet Ap$$

ES sets are the set-valued counterpart to ES states p in the sense that p is an ES state in the sense of Maynard Smith (1974) if, and only if, the singleton $\{p\}$ is an ES set.

A systematic investigation of the flows under RD shows that not every subset consisting of neutrally stable states is an ES set. Furthermore, there are attracting sets of fixed points which are not ES sets.

Specifically, we list that under matrix A and RD:

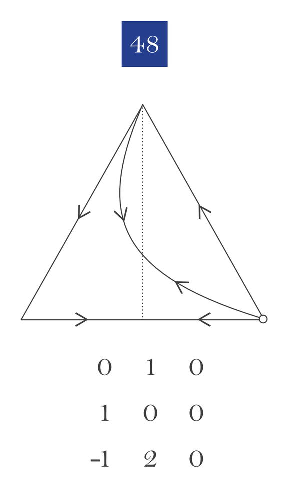
(a) *PP has ES sets with non-singleton components:* [21]; –[24]; –[29]; and [47].

(b) *PP has attracting sets (or singletons) that are not ES:* [5]: $P=\{p : p = ½\}$ is attractive but no ES set, since no $p \in P$ is neutrally stable; [9], [15]: $p=[⅓, ⅓, ⅓]$ is no ES state, but globally attracting; [12]: this is Zeeman's (1981) counterexample; [26]: $P=\{p : p_3 = 0\}$ is attractive, consists of NSSs, but is no ES set; –[30]: $P=\{p : p_3 = 0\}$ is attractive, but no ES set, since it contains no NSSs except [1, 0, 0].

(c) *PP has NSS that do not belong to ES sets:* [18], [19], –[20]: every p with $p_2=0$ is NS, but there is no ES set, since there is not asymptotically stable set; –[18], –[19], [20], [27]: as 18 but with p_1 instread of p_2; [22], –[23], [28], [33]: every p with $p_3=0$ and $p_1>0$ is NS, but there is no ES set, since there is no asymptotically stable set; –[22], [23], –[28], –[33]: as [22] but with p_2 instead of p_1; –[30]: see (b).

(d) *PP has Lyapunov stable states occur that do not belong to ES sets:* [3], (–)[28], (–)[33]: $p=[½, ½, 0]$ is Lyapunov stable, but is not NS; [13]: as with [3] but with $p=[⅗, ⅕, ⅕]$; [5], [9], [12], [15]: see (b); [6]: all p with $p_1 = ½$ and $p_2 > ¼$ are Lyapunov stable, but neither of these states is NS; [48]: all p with $p_1=p_2>0$ are Lyapunov stable, but only those with $⅓ \leq p_1 \leq ½$ are also NS.

The notation (–)[k] means that PP [k] and also PP –[k] obtained by flow reversal from [k] belong to this class.



Key

Sources are represented by open dots ◯; *sinks* by full dots ●; *centers* by dashed dots ⁘; and saddles by their insets and outsets (stable and unstable manifolds). In the *robust flows*, denoted by R, every orbit not specified flows from a source to a sink (except [17], where any non-constant orbit approaches any point on the edge infinitely often). In all other cases, some representative orbits are drawn.

Citations

Maynard Smith J (1974) The theory of games and the evolution of animal conflict. J Theor Biol **47**:209-221

Taylor P, and Jonker L (1978) Evolutionarily stable strategies and game dynamics. Math Biosci **40**:145-156

Thomas B (1985) On evolutionarily stable sets. J Math Biol 22:105-115

Zeeman EC (1981) Dynamics of the evolution of animal conflicts. J Theor Biol **89**:249-270.

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