The Limitations of Equilibrium Concepts in Evolutionary Game Theory

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Vocab

Evolutionary Game Theory: A mathematical framework for modeling evolutionary processes in biology and economics.

Equilibrium Concepts: A class of standard tools used, in this context, to predict the outcomes of evolutionary processes.

Motivation

At least **two reasons** why understanding the limitations of equilibrium concepts is important:

1. Equilibrium concepts are ubiquitous in a variety of disciplines in the social and biological sciences.

2. An evolutionary story has been told as justification for the predictions of traditional rational choice game theory [Sugden, 2001].

Motivation

- In certain disciplines, it is not uncommon to find the exclusive use of the equilibrium concepts, the leading candidate among which is the still the ESS.

- A search for "Ecology" and "ESS" in Google Scholar reveals that 13 of the top 20 papers since 2013 that used the method of ESS analysis used it exclusively—they followed what Huttegger and Zollman have termed "ESS methodology" [2014].

Aim

1. Make explicit what it means for equilibrium concepts to fail or succeed in the first place.

2. Circumscribe precisely when and why equilibrium concepts make errors of omission & commission.

Anatomy of evolutionary games

Evolutionary Game

1. Game

The interaction structure of the population.

In conflict, cooperation, signaling, mating, and so on.

2. Dynamics

Our hypothesis as to the behavior of evolution:

The nature of selection, mutation, drift, population size & structure, details of transmission, and exogenous factors.

Part 1: Introducing a game

Example

2-player 3x3 symmetric game

Column Player

Row Player

	a	b	c
a	1	1	1
b	1	1	1
c	2	2	1

Behaviors (phenotypes)
{a, b, c}

Payoffs (in reproductive success)

Part 2: Introducing an evolutionary dynamic

Column Player

Row Player

	a	b	c
a	1	1	1
b	1	1	1
c	2	2	1

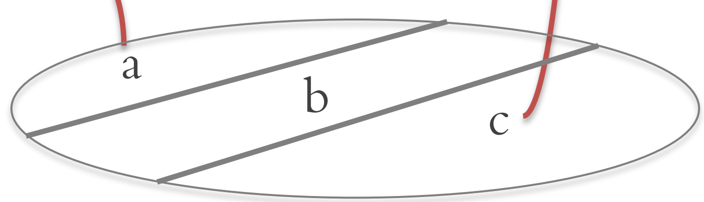
Choose a member of the population at random to interact.

This yields mean reproductive success of each phenotype.

$$u_c = P_a(2) + P_b(2) + P_c(1)$$

Leading idea

Fitness becomes frequency dependent.



Population (unit mass)

Population size is updated, and the process is repeated.

The replicator dynamics (RD)

Formulated as a continuous process

$$\frac{dP_i}{dt} = P_i(t)\left[u(i,P) - u(P,P)\right] \text{ for each } i = 1,\ldots,n$$

If a strategy is more fit than the population average,
then its proportion grows.

If it is less fit than the average, then its proportion shrinks.

Stationary points

For continuous RD

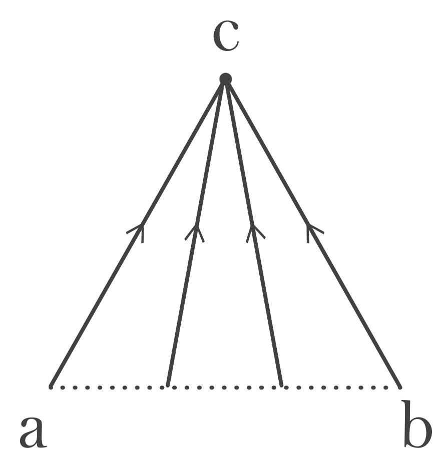
$$\frac{dP_i}{dt} = 0 \quad \text{for all} \quad i$$

Equilibria will be a subset of the stationary points of the dynamics.

A game under the replicator dynamics

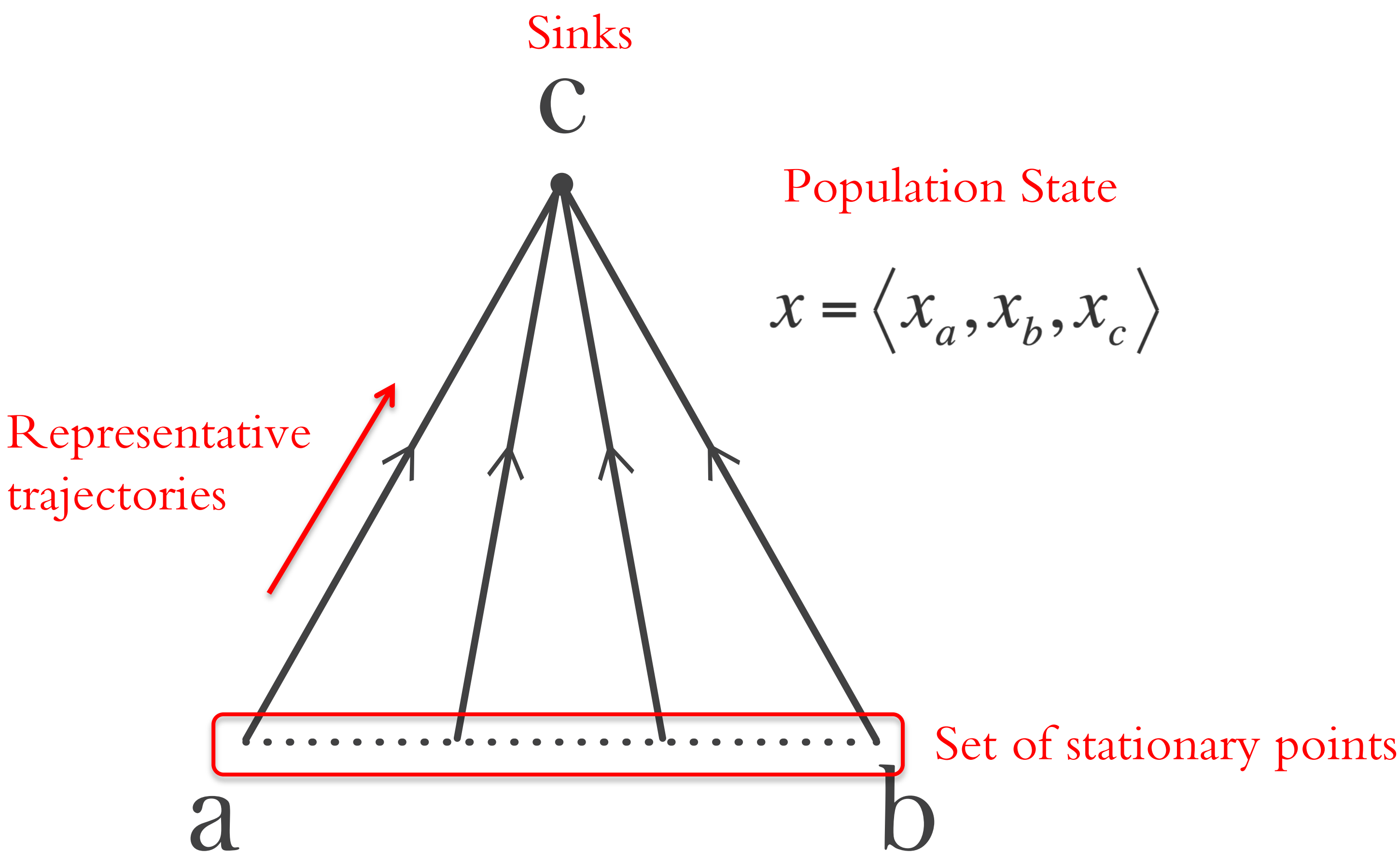
Recall the simple game from before

	a	b	c
a	1	1	1
b	1	1	1
c	2	2	1



Introducing the replicator dynamics induces a flow on the state space.

Phase portrait key



Equilibrium Concepts

First, we should note that the machinery we just introduced—an explicit articulation of the behavior of evolution (by a dynamics)—was not typical of traditional game theoretic analysis of evolutionary process.

Leading Idea

Equilibrium concept attempt to the infer the outcome of evolution exclusively from the interaction structure.

Formal definition of an equilibrium concept

More generally, an equilibrium concept is a mapping F from the set of all games Γ to the union of the states spaces Δ of all possible strategy profiles.

$$F : \Gamma \rightarrow \bigcup_{G \in \Gamma} 2^{\Delta_G}$$

So, for example, for the Nash equilibrium concept

$$F_{NE}(G) = \Delta_{NE} \subseteq \Delta_G$$

The Nash equilibrium (NE)

Column Player

Row Player

	a	b	c
a	1	1	1
b	1	1	1
c	2	2	1

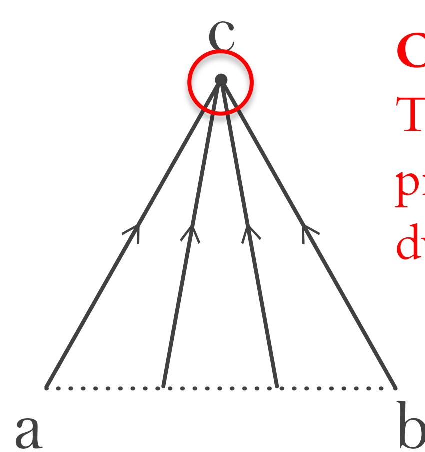
Nash equilibrium
(c , c)

Intuitively
A NE is a state in which no individual stands to gain by switching her strategy when all other players keep their strategies fixed.

A game under the replicator dynamics

We can return to the phase portrait to see how our Nash prediction aligns with the actual behavior of the dynamics.

	a	b	c
a	1	1	1
b	1	1	1
c	2	2	1



Outcome
The NE, here correctly predicts where the dynamics will end up.

We translate the Nash state into population proportions.

$$(c,c) \mapsto \langle 0,0,1 \rangle$$

Why equilibrium concepts?

Evolutionary processes can be thought of as solving optimization problems in a way not entirely dissimilar to the recommendations made to rational agents in classical game theory.

The search for an equilibrium concept in EGT has been driven by the hopes for an effective, general, and theoretically simple method of analysis.

The Problem

There are a breadth of conditions under which all equilibrium concepts fail to predict the correct outcome of the evolutionary dynamics.

Candidate equilibrium concepts

Nash Equilibrium

Strict Nash Equilibrium

Evolutionarily Stable Strategy (ESS)

Neutrally Stable Strategy (NSS)

Evolutionarily Stable Set (NSSet)

What does it mean for an equilibrium concept to succeed?

Often explained in terms of *evolutionary significance*. That is, for an equilibrium concept to be successful it must capture all and only the evolutionarily significant outcomes of a process [Huttegger & Zollman, 2010].

Evolutionary significance has been used a number of times in the EGT literature [Skyrms, 2000; Zollman et al, 2012; Huttegger et al, 2014], but has never been explicitly defined.

Evolutionary significance

Definition (Evolutionary significance)

In an evolutionary game, an *outcome* is *evolutionarily significant* if, and only if, it is a probable outcome of evolution.

Evolutionary significance

Definition (Evolutionary significance)

An outcome is *evolutionarily significant* if, and only if, given a uniform (or continuous, or exponential) probability distribution on initial conditions of the state space, a significant (non-measure-zero) set of initial conditions converge to the outcome in the limit, where they are stable in the face of arbitrarily small perturbations.

Mnemonic: Evolution must *get there*, and *stay there*.

Definition (Outcome)

Any long run regularity in the behavior of the process.

Evolutionary significance

Definition (Outcome)

Any long run regularity in the behavior of the process.

But then, to account for cycles and chaotic attractors, we may want to speak in terms of path stability.

1	0	2
2	1	0
0	2	1

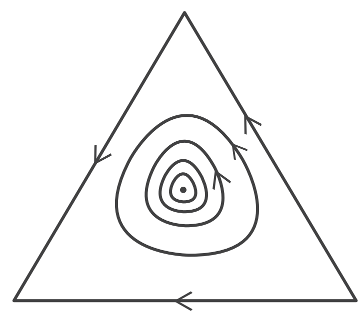
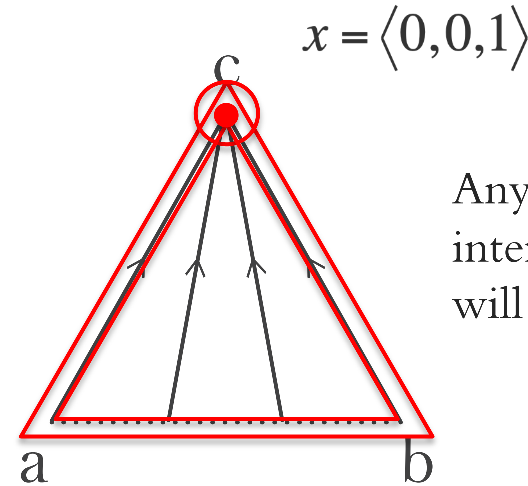


Figure 3: RPS Game

Evolutionary significance

We can apply our definition to say why the all-c state is the sole EvSig outcome of the dynamics.

	a	b	c
a	1	1	1
b	1	1	1
c	2	2	1



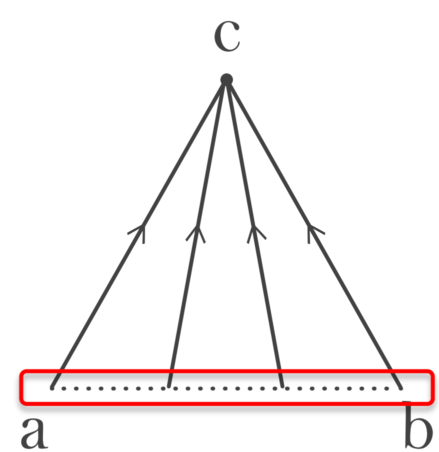
Any state on the interior of the simplex will converge to x.

Assume a uniform (or continuous, or exponential) distribution over states being the initial conditions of the dynamics.

Evolutionary significance

Now, let's consider an improbable outcome: The edge $\overline{ab}$

	a	b	c
a	1	1	1
b	1	1	1
c	2	2	1



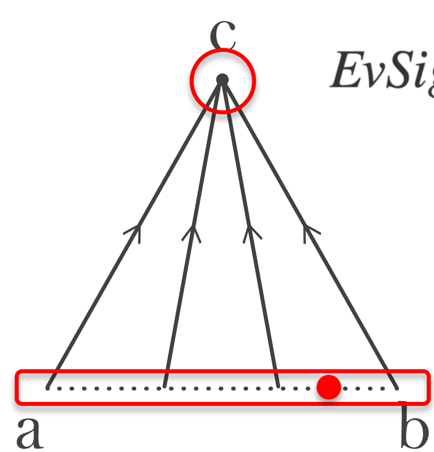
$$u_a = xu(a,a) + (1-x)u(a,b) = 1 = xu(b,a) + (1-x)u(b,b) = u_b$$

The chances of starting on this edge are negligible:
any one-dimensional edge composes a measure-zero subset of a two-dimensional space.

Evolutionary significance

But what if we know that the process will start at $\overline{ab}$?

	a	b	c
a	1	1	1
b	1	1	1
c	2	2	1



$$EvSig = \{\langle 0,0,1 \rangle\}$$

No state on this edge is stable under small perturbations.

Perhaps surprisingly, our conclusion remains:
No subset of the edge is a probable outcome of evolution.

(Relativized) Evolutionary significance

We can speak of *EvSig* with respect to (distributions over) sets of initial conditions.

If **initial conditions** compose a significant subset of the state space (are **mixed**), then the set of *RelEvSig* states is a subset of the the *EvSig* states.

If **initial conditions** compose a measure-0 subset of the state space (are **not mixed**), then the set of *RelEvSig* states is a subset of the the *Lyapunov stable* states.

Stability Concept

We provided a sharper definition of evolutionary significance so that we can connect it with precise mathematical precise formulations that can be used to assess the success of our equilibrium concepts.

Definition (Lyapunov stable) A state is *stable* if points near it remain near it.

Definition (Attracting) A state is *attracting* if nearby points tend toward it.

Definition (Asymptotically stable state) A state is *asymptotically stable* if it is both stable and attracting.

Assessing the success of equilibrium concepts

To Recap

We've given an explicit definition of what we're trying to capture with equilibrium concepts—we've delivered a precise notion of evolutionary significance—and we've hooked it up to the the mathematical machinery that we need to use.

Assessing the success of equilibrium concepts

This leaves us with some nice clarity as to what's going on:

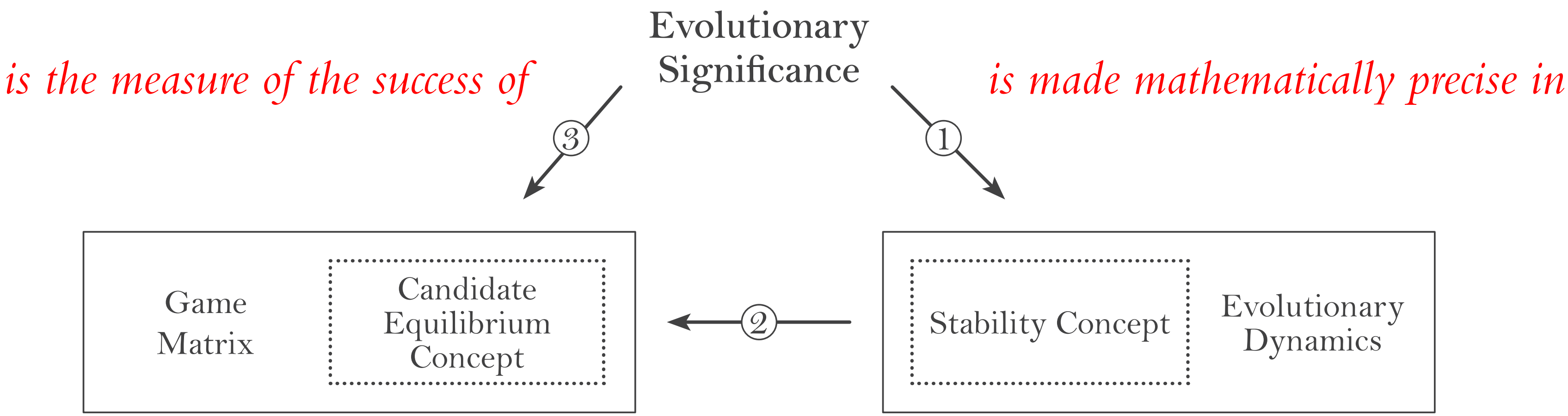


Figure 4: Assessment Diagram

stands in for evolutionary significance
in order to evaluate the success of

This allows assessment across different dynamics.

Two arguments against equilibrium concepts

Argument 1: Mathematical demonstrations of the limitations of each of the primary candidate equilibrium concepts.

Argument 2: In-principle limitations common to all of the candidate equilibrium concepts.

Methodology of Results

Our assumptions provide a kind of best-case scenario for the equilibrium concepts, so failure under these assumptions provides a strong argument against them.

- Justification for the replicator dynamics: The standard deterministic dynamics; designed specifically for the ESS concept; makes the same assumptions about populations.

- Justification for asymptotic stability: Aligns with our definition of evolutionary significance; the EC's fair strictly worse under Lyapunov stability.

Mathematical demonstrations

Nash refinements

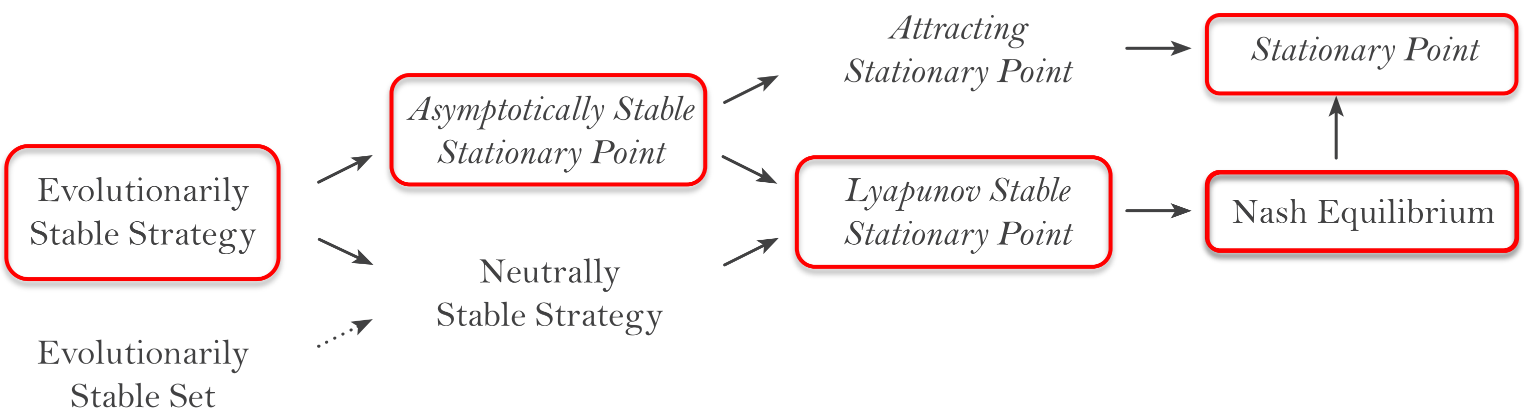


FIGURE 5: LOGICAL RELATIONS BETWEEN EQUILIBRIUM AND STABILITY CONCEPTS

A natural place to begin our examination is at the NE: In their logical structure, all other equilibrium concepts (except the ESSet) are refinements of the NE concept.

NB: All converse implications are typically not true.

Nash equilibrium

Definition (Nash Equilibrium)

A state $x^* \in \Delta$ is a *Nash equilibrium* if,

for all i s.t. $x_i \in \Delta$, $u\left(x_i^*, x_{-i}^*\right) \geq u\left(x_i, x_{-i}^*\right)$.

Nash equilibrium

Definition (Nash Equilibrium)

A strategy $x \in \Delta$ is a *Nash equilibrium* if,

for all $y \in \Delta$, $u(x,x) \geq u(y,x)$.

TABLE 1: **Equilibrium Concepts *for* Replicator Dynamics**

	Too Weak Captures some outcomes that ARE NOT evolutionarily significant that are:			**Too Strong** Fails to capture some outcomes that ARE evolutionarily significant that are:		
	Unstable	Not Attracting	Small Basins of Attraction	Sets of States (as Sets)	Asymptotically Stable States	Cycles
Nash Equilibrium (NE)	☑ *Figure 6*	☑ *Figure 6*	☑ *Figure 9*	☑ *Figure 12*	☐	☑ *Figure 3*
Strict Nash Equilibrium (Strict NE)	☐	☐	☑ *Figure 9*	☑ *Figure 12*	☑ *Figures 8, 10, 11*	☑ *Figure 3*
Evolutionarily Stable Strategy (ESS)	☐	☐	☑ *Figure 9*	☑ *Figure 12*	☑ *Figures 10, 11*	☑ *Figure 3*
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The limits of Nash equilibrium

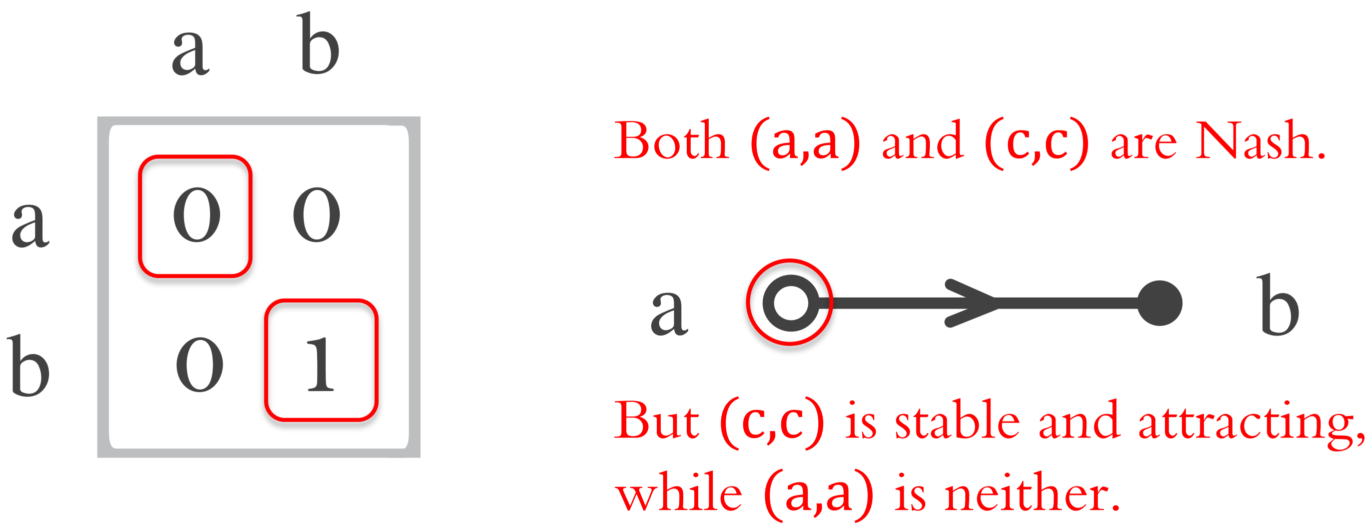

Figure 6: Unstable Nash Equilibrium

Strict Nash equilibrium

Definition (strict Nash equilibrium)

A state $x \in \Delta$ is a *strict Nash equilibrium* if,

for all $y \in \Delta$, s.t. $y \neq x$, $u(x,x) > u(y,x)$.

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The limits of strict Nash equilibrium

	H	D
H	0	3
D	1	2

$$x^* = \left\langle \tfrac{1}{2}, \tfrac{1}{2} \right\rangle$$

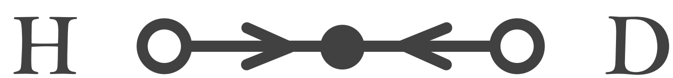

Figure 8: Hawk-Dove Game

The Nash refinement project

Nash equilibrium — **Too weak** *(includes unstable states)*

Neutrally stable strategy (NSS)
Evolutionarily stable strategy (ESS)
Evolutionarily stable set (ESSet)

} Can we find success in the middle ground?

Strict Nash equilibrium — **Too strong** *(excludes stable mixed states)*

Evolutionarily stable strategy

Definition (Evolutionarily stable strategy)

A state $x \in \Delta$ is an *evolutionarily stable strategy* if,

for all $y \in \Delta$ s.t. $y \neq x$,

(1) $u(x,x) \geq u(y,x)$, and

(2) if $u(x,x) = u(y,x)$, then $u(x,y) > u(y,y)$.

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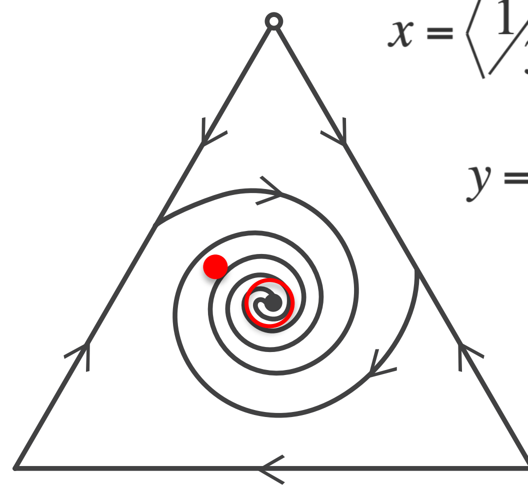
Elliptical attractors

1	1	1
0	0	3
2	1	0



$x = \langle 1/3, 1/3, 1/3 \rangle$

$y = \langle 3/8, 1/4, 3/8 \rangle$

Figure 10: Globally Attracting Non-ES State

✓ (1) $u(y,x) = 1 = u(x,x)$

✗ (2) $u(y,y) = 1.03125 > 0.95833 = u(x,y)$

The in-principle argument

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Small basins of attraction

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & \varepsilon \\ 0 & 0 & 2\varepsilon \end{pmatrix}$$

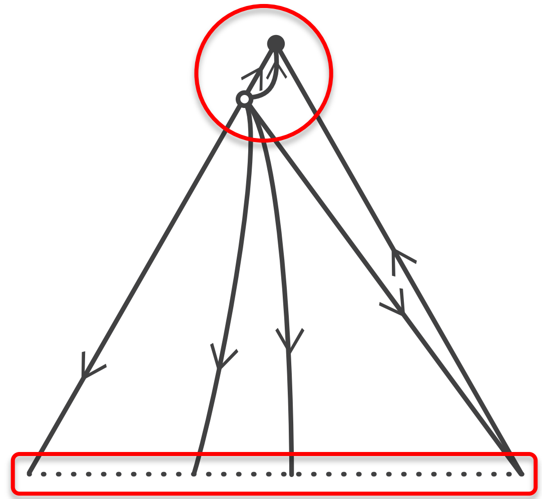

FIGURE 9: SMALL BASIN OF ATTRACTION

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Evolutionarily Stable Set (ESSet)	☐	☐	☑ *Figure 9*	☐	☑ *Figures 10, 11*	☑ *Figure 3*

Cycles

1	0	2
2	1	0
0	2	1

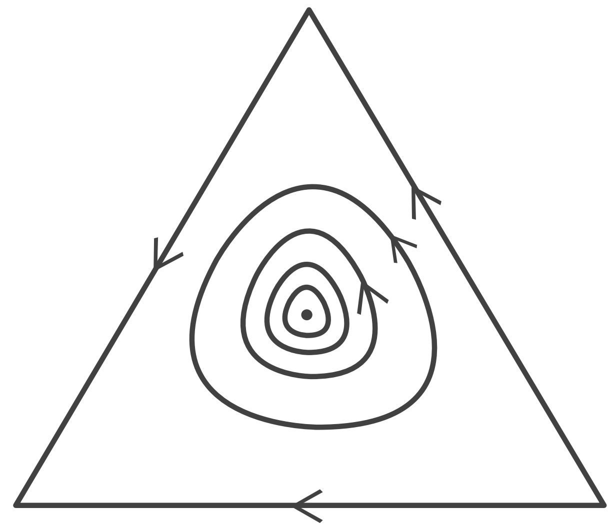

FIGURE 3: RPS GAME

Have been found to occur in nature [Sinervo et al, 1996].

Chaotic behavior

1.05	0	0	1.5
2.05	0.6	0	0
0	0.4	0	2.25
1.55	0.5	0.1	0.5

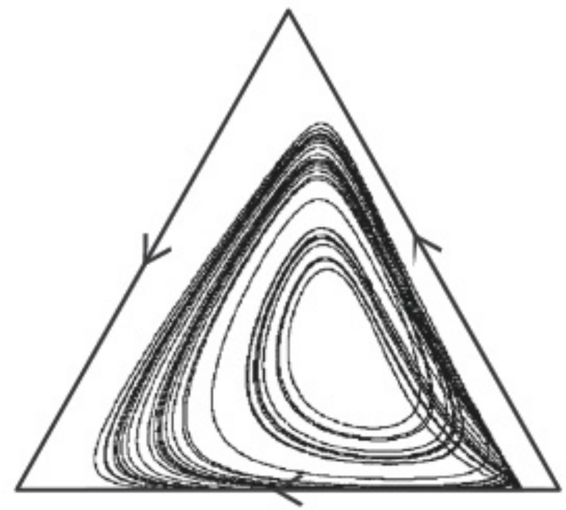

FIGURE 13: A STRANGE ATTRACTOR

See (Skyrms, [1992]).

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Limitations at different complexity classes

2x2 Games	✓
3x3 Games	*Elliptical attractors, cycles, AS sets*
4x4 Games	*Chaotic behavior*

Limitations under different assumptions

Lyapunov Stability
(Stability Concept)

ECs (except, possibly, the NSS) fair strictly worse—see paper.

Stochastic Dynamics
(Model of Evolution)

ECs fair worse—see my newer work.

Summary

- We have provided a novel account of 'evolutionary significance'.
- Demonstrated that, even under ideally favorable assumptions, each of the primary candidate equilibrium concepts is simultaneously too strong and too weak.
- Presented an in-principle argument that there are EvSig outcomes that cannot be effectively be expressed, much less predicted, by equilibrium concepts.

Our Moral

Equilibrium concepts are typically going to be unreliable tools for the analysis of dynamic evolutionary processes, and that a more complicated, but a more interesting picture emerges from explicit investigation of the underlying dynamics.

Next steps

Analyze the conditions for agreement and divergence between our main deterministic and stochastic models of evolution.

Exploration of the classes of games under which strong mutation provides a sufficient condition for agreement between the MPM and RD constitutes an interesting subject for future study

Thank you.